

A GENERALIZATION OF A RESULT OF DONG AND SANTOS-STURMFELS ON THE ALEXANDER DUAL OF SPHERES AND BALLS

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ABSTRACT. We prove a generalization of a result by Dong and Santos-Sturmfels about the homotopy type of the Alexander dual of balls and spheres. Our results involve NH -manifolds, which were recently introduced as the non-homogeneous (or non-pure) counterpart of classical polyhedral manifolds. We show that the Alexander dual of an NH -ball is contractible and the Alexander dual of an NH -sphere is homotopy equivalent to a sphere. We also prove that NH -balls and NH -spheres arise naturally as the double duals of standard balls and spheres.

1. INTRODUCTION

Let K be a finite simplicial complex. Fix a ground set of vertices V which contains the set V_K of vertices of K , and let Δ denote the simplex spanned by V . The classical Alexander duality theorem admits a combinatorial formulation in terms of a simplicial homotopy representative K^{*v} of $|\partial\Delta| - |K|$ called the Alexander dual of K . In this form the theorem asserts that $H_i(K) \simeq H^{n-i-3}(K^{*v})$, where n is the cardinal of V and both homology and cohomology groups are reduced (see [14, Theorem 71.1] and [2]). In light of this result, it is natural to ask if the homotopy type of K can also be deduced from K^{*v} . Unfortunately this is generally not the case. There are canonical examples of contractible polyhedra and homotopy spheres whose Alexander duals are not respectively contractible or homotopy equivalent to spheres. Moreover, it can be shown that for any finitely presented group G there is a finite simply connected complex K such that $\pi_1(K^{*v}) = G$ (see [13]). In 2002, Dong [6] proved that the Alexander dual of a simplicial sphere has again the homotopy type of a sphere. One year later Santos and Sturmfels [15] showed that the Alexander duals of simplicial balls are contractible spaces. Dong's approach relies mainly on convexity and Santos-Sturmfels' proof uses Dong's result on spheres. Both results evidence that a locally well-behaved structure on the complex forces homotopy stableness on its dual and one may ask whether other manifold-like constructions can hold similar properties.

The NH -manifolds are natural candidates for this. These complexes were recently introduced in [5] as a generalization of combinatorial manifolds to the non-homogeneous setting (or, more precisely, to the non-necessarily homogeneous setting). The study of NH -manifolds was in part motivated by the theory of non-pure shellability due to Björner and Wachs [3]. It was also shown in [5] that they appear when investigating Pachner moves between manifolds: if two polyhedral manifolds (with or without boundary) are

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PL-homeomorphic then they are related by a finite sequence of factorizations involving NH -manifolds.

In this paper we prove a generalization of the results of Dong and Santos-Sturmfels to non-necessarily homogeneous balls and spheres. Concretely, we show that the Alexander dual of an NH -ball is a contractible space and the Alexander dual of an NH -sphere is homotopy equivalent to a sphere (Theorems 4.11 and 4.12 below). These results extend considerably the previous ones. Our approach is based on the local structure of these polyhedra and, as a by-product, we exhibit an alternative and simple proof of the original results which relies on the local nature of the manifolds, in contrast to the previous treatments.

The second aim of this article is to use the theory of NH -manifolds to characterize the (topological, simplicial) structure of the Alexander duals of balls and spheres in the following sense. Given a subspace A of the d -sphere S^d , since the complement $B = S^d - A$ is also a subspace of $S^{d'}$ for any $d' \geq d$, it is natural to study the relationship between A and $S^{d'} - B$ (the complement of its complement in a sphere of higher dimension). In the combinatorial setting, this amounts to understand the *double dual* $L = (K^{*V})^{*_{V'}}$ where $V_K \subseteq V \subsetneq V'$. We prove that the double duals of balls (resp. spheres) are NH -balls (resp. NH -spheres).

The rest of the paper is organized as follows. In section two we recall the basic properties of classical combinatorial manifolds and NH -manifolds and prove a result on the existence of spines for NH -manifolds with boundary. This result is used in the proofs of the main theorems but it is also interesting in its own right.

In section three we compare the Alexander duals of a complex with respect to different ground sets of vertices and characterize the Alexander double duals of balls and spheres. As a corollary we show that NH -balls and NH -spheres appear as double duals of (classical) balls and spheres.

In the last section of the article we prove the generalization of Dong's and Santos-Sturmfels' results on the Alexander dual of spheres and balls to the non-homogeneous setting.

2. PRELIMINARIES

2.1. Basic notations. All complexes considered in this paper are finite. Given a set of vertices V , $|V|$ will denote its cardinality and $\Delta(V)$ the simplex spanned by the vertices of V . $\Delta^d = \Delta(\{0, \dots, d\})$ will denote a generic d -simplex and $\partial\Delta^d$ its boundary. The set of vertices of a complex K will be denoted V_K and we set $\Delta_K = \Delta(V_K)$. We write $\sigma < \tau$ when σ is a face of τ and $\sigma \prec \tau$ when it is an immediate face. A simplex is *maximal* or *principal* in a complex K if it is not a proper face of any other simplex of K . A *ridge* of K is an immediate face of a principal simplex. Two simplices $\sigma, \tau \in K$ are *adjacent* if $\sigma \cap \tau$ is an immediate face of σ or τ .

As usual $K * L$ will denote the join of the complexes K and L . By convention, if \emptyset is the empty simplex and $\{\emptyset\}$ the complex containing only the empty simplex then $K * \{\emptyset\} = K$ and $K * \emptyset = \emptyset$. For $\sigma \in K$, $lk(\sigma, K) = \{\tau \in K : \tau \cap \sigma = \emptyset, \tau * \sigma \in K\}$ denotes its *link* and $st(\sigma, K) = \sigma * lk(\sigma, K)$ its (closed) *star*. The union of two complexes K, L will be denoted by $K + L$. A subcomplex $L \subset K$ is said to be *top generated* if every principal simplex of L is also principal in K .

We write $K \simeq_{PL} L$ when K and L are PL-homeomorphic; that is, whenever they have a subdivision in common. We shall frequently identify a complex K with its geometric realization and we shall write $K \simeq L$ when K is homotopy equivalent to L .

Given $t \geq 0$, $\Sigma^t K = \partial\Delta^t * K$ will denote the simplicial t -fold (unreduced) suspension of K .

A principal simplex $\tau \in K$ is *collapsible* in K if there is a ridge $\sigma \prec \tau$ which is not a face of any other simplex of K (i.e. σ is a free face). If τ is collapsible, the operation which transforms K into $K - \{\tau, \sigma\}$ is called an *elementary (simplicial) collapse* and is denoted by $K \searrow^e K - \{\tau, \sigma\}$. It is easy to see that $K - \{\tau, \sigma\} \subset K$ is a strong deformation retract. The inverse operation is called an *elementary (simplicial) expansion*. If there is a sequence $K \searrow^e K_1 \searrow^e \cdots \searrow^e L$ we say that K *collapses* to L (or equivalently, L *expands* to K) and write $K \searrow L$ or $L \nearrow K$. A complex K is *collapsible* if it has a subdivision which collapses to a single vertex.

2.2. Combinatorial manifolds. We recall some basic definitions and properties of the classical theory of combinatorial manifolds. For a comprehensive exposition of the subject we refer the reader to [8, 9, 11].

A combinatorial d -ball is a complex which is PL-homeomorphic to Δ^d . A combinatorial d -sphere is a complex PL-homeomorphic to $\partial\Delta^{d+1}$. By convention, $\{\emptyset\} = \partial\Delta^0$ is considered a sphere of dimension -1 . A combinatorial d -manifold is a complex M such that $lk(v, M)$ is a combinatorial $(d-1)$ -ball or $(d-1)$ -sphere for every $v \in V_M$. It is easy to verify that d -manifolds are homogeneous complexes of dimension d ; that is, all of its principal simplices are d -dimensional. The link of any simplex in a manifold is also a ball or a sphere and the class of combinatorial manifolds is closed under PL-homeomorphisms. In particular, combinatorial balls and spheres are combinatorial manifolds. By a result of J.H.C. Whitehead, combinatorial d -balls are precisely the collapsible combinatorial d -manifolds (see [8, Corollaries III.6 and III.17]) and by a result of Newman, if S is a combinatorial d -sphere containing a combinatorial d -ball B , then the closure $\overline{S - B}$ is a combinatorial d -ball (see [8, 9, 11]).

The boundary ∂M of a combinatorial d -manifold M can be regarded as the set of simplices whose links are combinatorial balls. This coincides with the usual definition of boundary for d -homogeneous complexes as the subcomplex generated by the mod 2 sum of the $(d-1)$ -simplices. It is easy to see that ∂M is a $(d-1)$ -combinatorial manifold without boundary.

A *weak d -pseudomanifold* without boundary is a d -homogeneous simplicial complex P satisfying that each $(d-1)$ -simplex is contained in exactly two d -simplices. It is easy to see that in this case $lk(\sigma, P)$ is a weak $(d - \dim(\sigma) - 1)$ -pseudomanifold for every $\sigma \in P$ and that $H_d(P; \mathbb{Z}_2) \neq 0$, since the mod 2 sum of the d -simplices of P is a generating cycle. A *d -pseudomanifold* is a weak d -pseudomanifold with or without boundary (i.e. the $(d-1)$ -simplices are contained in *at most* two d -simplices) which is strongly connected; that is, any two d -simplices σ, τ can be connected by a sequence of d -simplices $\sigma = \eta_0, \dots, \eta_k = \tau$ such that $\eta_i \cap \eta_{i+1}$ is $(d-1)$ -dimensional for each $i = 0, \dots, k-1$ (i.e. η_i and η_{i+1} are adjacent). It is easy to see that a connected combinatorial d -manifold is a d -pseudomanifold.

2.3. Non-homogeneous manifolds. *NH*-manifolds are the non-homogeneous versions of combinatorial manifolds and play a key role in this work. We give next a brief summary of the subject and refer the reader to [5] for a more detailed exposition.

NH -manifolds have a local structure consisting of Euclidean spaces of varying dimensions. In Figure 1 we exhibit some examples of NH -manifolds.

Definition 2.1. An NH -manifold (resp. NH -ball, NH -sphere) of dimension 0 is a manifold (resp. ball, sphere) of dimension 0. An NH -sphere of dimension -1 is, by convention, the empty set. For $d \geq 1$, we define by induction

- An NH -manifold of dimension d is a complex M of dimension d such that $lk(v, M)$ is an NH -ball of dimension $0 \leq k \leq d - 1$ or an NH -sphere of dimension $-1 \leq k \leq d - 1$ for all $v \in V_M$.
- An NH -ball of dimension d is a collapsible NH -manifold of dimension d .
- An NH -sphere of dimension d and homotopy dimension k is an NH -manifold S of dimension d such that there exist a top generated NH -ball B of dimension d and a top generated combinatorial k -ball L such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B + L$ is a *decomposition* of S and write $\dim_h(S)$ for the homotopy dimension of S .

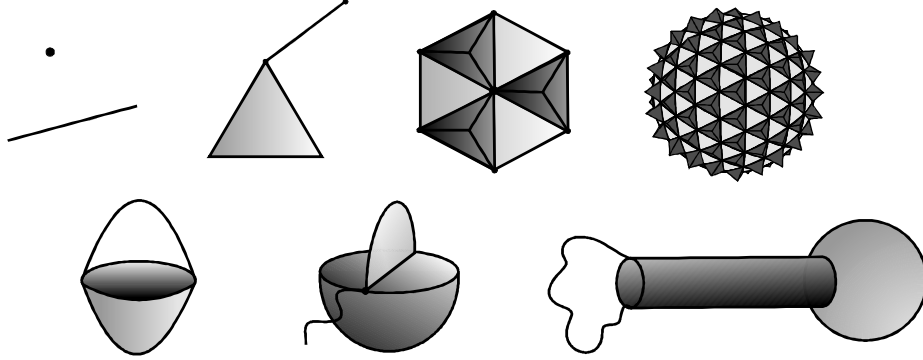


Figure 1. Examples of NH -manifolds. The first, fourth and fifth figures are NH -spheres of dimension 1, 3 and 2 and homotopy dimension 0, 2 and 1 respectively. The second, third and sixth figures are NH -balls. The last figure is an NH -bouquet.

In [5] it is proved that NH -manifolds satisfy many (generalized) results of the classical theory of combinatorial manifolds. Also, by [5, Theorem 3.6], homogeneous NH -manifolds are standard combinatorial manifolds. We next summarize the relevant results of this theory that will be used in this article.

Theorem 2.2. Let M be an NH -manifold of dimension d , let $\sigma \in M$ and let B_1, B_2 be NH -balls and S_1, S_2 be NH -spheres.

- (1) $lk(\sigma, M)$ is an NH -ball or an NH -sphere.
- (2) NH -manifolds, NH -balls and NH -spheres are closed under PL -homeomorphisms.
- (3) $B_1 * B_2$ and $B_1 * S_2$ are NH -balls. $S_1 * S_2$ is an NH -sphere.
- (4) If M is connected then it is an NH -pseudomanifold; i.e. (a) for each ridge $\sigma \in M$, $lk(\sigma, M)$ is either a point or an NH -sphere of homotopy dimension 0; and (b) given any two principal simplices $\sigma, \tau \in M$, there exists a sequence $\sigma = \eta_1, \dots, \eta_s = \tau$ of principal simplices of M such that η_i and η_{i+1} are adjacent for each $1 \leq i \leq s - 1$.

The *pseudoboundary* $\tilde{\partial}M$ of an NH -manifold is the set of simplices whose links are NH -balls (this is not in general a simplicial complex). The boundary ∂M is the simplicial complex generated by the simplices in $\tilde{\partial}M$. By [5, Proposition 4.3], $\tilde{\partial}M$ is a complex if and only if M is homogeneous. This implies that boundaryless NH -manifolds are classical (combinatorial) manifolds. This shows that, unlike classical manifolds, NH -spheres which are non-homogeneous do have boundary. By [5, Lemma 4.8] if $S = B + L$ is a decomposition of an NH -sphere then $lk(\sigma, S)$ is an NH -sphere with decomposition $lk(\sigma, B) + lk(\sigma, L)$ for every $\sigma \in L$.

Given an NH -manifold M , we denote by $\overset{\circ}{M}$ the relative interior of M , which is the set of simplices whose links are NH -spheres (of any dimension).

The following is a special case of [5, Theorem 6.3], which is a generalization of the classical Alexander's theorem on regular expansions (see [11, Theorem 3.9]).

Theorem 2.3. *Let M be an NH -ball (resp. NH -sphere) and B a combinatorial ball. Suppose $M \cap B \subseteq \partial B$ is an NH -ball or an NH -sphere generated by ridges of M or B and that $(M \cap B)^\circ \subseteq \tilde{\partial}M$. Then*

- (1) $M + B$ is an NH -ball (resp. NH -sphere) if $M \cap B$ is an NH -ball.
- (2) $M + B$ is an NH -sphere if $M \cap B$ is an NH -sphere.

The next two results, which are interesting in their own right, will be used in the last section of the article.

Proposition 2.4. *Let M be a connected NH -manifold of dimension d such that $H_d(M; \mathbb{Z}_2) \neq 0$. Then, M is a combinatorial d -manifold (without boundary). In particular, if S is an NH -sphere with $\dim_h(S) = \dim(S)$ then S is a combinatorial sphere.*

Proof. By [5, Theorem 3.6] it suffices to prove that M is homogeneous. Let c be a generating d -cycle of $H_d(M; \mathbb{Z}_2)$ and let $K \subset M$ be the subcomplex generated by the d -simplices appearing in c with nonzero coefficients. We shall show that $M = K$. Note that since $K \subset M$ is top generated and M is an NH -pseudomanifold (see Theorem 2.2 (4)) then K is a weak pseudomanifold without boundary (since c is a cycle). If $M \neq K$, let $\eta \in M - K$ be a principal simplex adjacent to K and set $\rho = \eta \cap K$. Since by dimensional considerations $\rho \prec \eta$ then $lk(\rho, M) = lk(\rho, M - \eta) + lk(\rho, \eta)$ is an NH -sphere of homotopy dimension 0. But $lk(\rho, K) \subset lk(\rho, M - \eta)$ is a weak pseudomanifold without boundary and hence $H_{\dim(lk(\rho, K))}(lk(\rho, K); \mathbb{Z}_2) \neq 0$. This contradicts the fact that $lk(\rho, M - \eta)$ is an NH -ball since a generating cycle in $lk(\rho, K)$ is also generating in $lk(\rho, M - \eta)$. Note also that $\partial M = \partial K = \emptyset$. \square

Corollary 2.5 (Existence of spines for NH -manifolds). *Every connected NH -manifold M with non-empty boundary has a spine (i.e. it collapses to a subcomplex of smaller dimension).*

Proof. Let d be the dimension of M and let Y^d be the d -homogeneous subcomplex of M (i.e. the subcomplex of M generated by the d -simplices). Start collapsing the d -simplices of Y^d and suppose we get stuck before depleting all the d -simplices. Then, there is a boundaryless d -pseudomanifold $L \subset Y^d \subset M$ and hence $0 \neq H_d(L; \mathbb{Z}_2) \subset H_d(M; \mathbb{Z}_2)$. By Proposition 2.4, M is a combinatorial manifold without boundary, which is a contradiction. \square

3. THE ALEXANDER DOUBLE DUAL OF BALLS AND SPHERES

3.1. The Alexander dual with respect to different ground sets of vertices. We first study the relationship between the Alexander dual of a complex relative to its own set of vertices and to a bigger ground set of vertices. This is a natural question since geometrically it amounts to analyze the relation between the complement of a complex when seen as subspace of spheres of different dimensions.

For a complex K we denote $K^* = \{\sigma \in \Delta_K \mid \sigma^{c_{V_K}} \notin K\}$ the Alexander dual *with respect to the ground set* V_K . Here $\sigma^{c_{V_K}} = \Delta(V_K - V_\sigma)$ is the *complement* of σ in V_K . Now, for a vertex set $V \supseteq V_K$ we consider the simplex $\tau = \Delta(V - V_K)$ and we denote the Alexander dual of K *relative to* V by K^τ ; that is, $K^\tau = \{\eta \in \Delta(V) \mid \eta^{c_V} \notin K\}$ where $\eta^{c_V} = \Delta(V - V_\eta)$ is the complement of η with respect to $V = V_K \cup V_\tau$. We will omit the subscript V_K or V in the complement when that is clear from the context. Note that if $\tau = \emptyset$ then $K^\tau = K^*$ is the Alexander dual of K relative to its own set of vertices.

We shall use the following convention regarding the Alexander dual of simplices and boundary of simplices: $(\Delta^d)^* = \emptyset$ and $(\partial\Delta^d)^* = \{\emptyset\}$.

Lemma 3.1. *Let K be a simplicial complex and let τ be a (non-empty) simplex disjoint from K . Then,*

$$(A) \quad K^\tau = \partial\tau * \Delta_K + \tau * K^*.$$

Here K^* is considered as a subcomplex of the simplex Δ_K .

In particular, we have the following consequences.

- (1) If K is not a simplex or $\dim(\tau) \geq 1$ then $V_{K^\tau} = V_K \cup V_\tau$. If $K = \eta$ is a simplex and $\dim(\tau) = 0$ then $\eta^\tau = \eta$. In any case, $V_K \subseteq V_{K^\tau}$.
- (2) If K is not a simplex or $\dim(\tau) \geq 1$ then $(K^\tau)^* = K$.
- (3) If $V_{K^*} \subsetneq V_K$ and $\rho = \Delta(V_K - V_{K^*})$ then $(K^*)^\rho = K$.
- (4) If K is not a simplex then $K^\tau \simeq \Sigma^t K^*$ for some $t \geq 0$.

Proof. Set $V = V_K \cup V_\tau$. Let $\sigma \in K^\tau$ be a principal simplex, so $\sigma^{c_V} \notin K$. If $\tau < \sigma$, say $\sigma = \tau * \eta$, then $\sigma^{c_V} = \eta^{c_{V_K}}$ and therefore $\sigma = \tau * \eta \in \tau * K^*$. Any other simplex in K^τ not containing τ lies trivially in $\partial\tau * \Delta_K$. For the other inclusion, if $\sigma = \tau * \eta$ is principal and $\eta \in K^*$ then $\sigma^{c_V} = \eta^{c_{V_K}} \notin K$, and hence $\sigma \in K^\tau$. If $\sigma \in \partial\tau * \Delta_K$ is principal then, in particular, $\Delta_K < \sigma$ and therefore $\sigma^{c_V} < \tau$. Since no vertex of τ lies in K , $\sigma^{c_V} \notin K$ and then $\sigma \in K^\tau$.

Item (1) follows directly from formula (A) and items (2)-(3) from the fact that for a fixed ground set V , $(K^{*V})^{*V} = K$. Finally, (4) follows from formula (A) since both summands are contractible (see Lemma 4.3 (1)). \square

Note that the equation in (4) also holds for $\tau = \emptyset$ taking $t = 0$.

Lemma 3.2. *Let K be a simplicial complex of dimension d that is not a d -simplex. The following statements are equivalent.*

- (1) $|V_K| = d + 2$.
- (2) $V_{K^*} \neq V_K$.
- (3) $K \neq K^{**}$.

Proof. Suppose that $|V_K| = d + 2$ and let $\sigma \in K$ be a d -simplex. Then the only vertex $v \in V_K - V_\sigma$ is not in V_{K^*} . Conversely, if $w \in V_K - V_{K^*}$ then $w^c \in K$. Since K is not a

d -simplex then $|V_K| \geq d + 2$. Since w^c is the simplex spanned by the vertices in $V_K - \{w\}$ and $\dim(K) = d$ then $|V_K| \leq d + 2$. This proves that (1) and (2) are equivalent.

(2) implies (3) since $V_{K^{**}} \subseteq V_{K^*} \subseteq V_K$. Also, (3) implies (2) since if $V_{K^*} = V_K$ then $K^{**} = K$. \square

Corollary 3.3. *Let K be a simplicial complex and let τ be a (non-empty) simplex disjoint from K . Then,*

- (1) *If K is not a simplex or $\dim(\tau) \geq 1$ then $|V_{K^\tau}| = \dim(K^\tau) + 2$.*
- (2) *The subcomplexes $\partial\tau * \Delta_{K, \tau} * K^* \subset K^\tau$ in formula (A) of Lemma 3.1 are top generated.*

Proof. If K is not a simplex, item (1) follows directly from Lemmas 3.1 and 3.2. If $K = \eta$ is a simplex and $\dim(\tau) \geq 1$ then $\eta^\tau = \partial\tau * \eta$ which has dimension $\dim(\tau) + \dim(\eta)$ and $\dim(\tau) + 1 + |V_\eta| = \dim(\tau) + \dim(\eta) + 2$ vertices.

For (2), simply notice that $\partial\tau * \Delta_K \cap \tau * K^* = \partial\tau * K^*$ and that K^* is always properly contained in Δ_K . \square

Remark 3.4. Lemma 3.1 and Corollary 3.3 state that every complex is the Alexander dual of a complex of dimension d and $d + 2$ vertices for some $d \geq 0$.

3.2. Alexander double duals of balls and spheres. Suppose A is a subspace of the d -sphere S^d . The complement $B = S^d - A$ is also a subspace of $S^{d'}$ for any $d' \geq d$ and taking into account that $S^d - B = A$ it is natural to ask what kind of relationship exists between A and $S^{d'} - B$. In the simplicial setting this amounts to understand the similarities between a complex K and $(K^\tau)^\sigma$ for $V_\tau \cap V_K = \emptyset$ and $V_\sigma \cap V_{K^\tau} = \emptyset$. We call the complex $(K^\tau)^\sigma$ a *double dual* of K . When $\tau = \sigma = \emptyset$ we call $(K^*)^* = K^{**}$ the *standard double dual* of K .

Double duals share many of the properties of the original complexes. For example, it is easy to see from formula (A) of Lemma 3.1 and Lemmas 3.1 and 3.2 that $(K^\tau)^\sigma \simeq \Sigma^t K$ for some $t \geq 0$ if $|V_K| \geq d + 3$. Also, it can be shown that a complex K is shellable if and only if $(K^\tau)^\sigma$ is shellable. We are mainly interested in double duals of combinatorial balls and spheres and we shall show that they are precisely the NH -balls and NH -spheres. The result basically follows from the following

Lemma 3.5. *Let K be a simplicial complex. If $V_K \subseteq V$ and $\eta \neq \emptyset$ is a simplex, then*

$$L = \partial\eta * \Delta(V) + \eta * K$$

is an NH -ball (resp. NH -sphere) if and only if K is an NH -ball (resp. NH -sphere). Here K is viewed as a subcomplex of the simplex $\Delta(V)$.

Proof. Put $\Delta = \Delta(V)$. If L is an NH -ball or NH -sphere then $K = lk(\eta, L)$ is either an NH -ball or NH -sphere by Theorem 2.2 (1). Since $\partial\eta * \Delta$ and $\eta * K$ are collapsible and $\partial\eta * \Delta \cap \eta * K = \partial\eta * K$ then K will be an NH -ball if L is one and an NH -sphere if L is one.

Suppose K is an NH -ball or NH -sphere. By Theorem 2.2 (3), $\partial\eta * \Delta$ is a combinatorial ball, $\eta * K$ is an NH -ball and $\partial\eta * \Delta \cap \eta * K = \partial\eta * K$ is an NH -ball or NH -sphere according to K . We use Theorem 2.3 to prove that L is an NH -ball or NH -sphere. Note that $\partial\eta * K$ is trivially contained in $\partial(\partial\eta * \Delta)$ and it is generated by ridges of $\eta * K$. Also, if $\rho \in (\partial\eta * K)^\circ$ and $\hat{\eta}$ denotes the barycenter of η then

$$lk(\rho, \eta * K) \simeq_{PL} lk(\rho, \hat{\eta} * \partial\eta * K) = \hat{\eta} * lk(\rho, \partial\eta * K)$$

which is an NH -ball by Theorem 2.2 (2). This implies that $\partial\eta * K \subset \tilde{\partial}(\eta * K)$. By Theorem 2.3, L is an NH -ball or NH -sphere. \square

Theorem 3.6. *Let K be a simplicial complex and let τ be a simplex (possibly empty) disjoint from K and σ a simplex (possibly empty) disjoint from K^τ . Then K is an NH -ball (resp. NH -sphere) if and only if $(K^\tau)^\sigma$ is an NH -ball (resp. NH -sphere).*

Proof. We first prove the case $\tau = \sigma = \emptyset$. By Lemma 3.2 we may assume $|V_K| = \dim(K) + 2$. Let $\rho = \Delta(V_K - V_{K^*}) \neq \emptyset$ so $K = (K^*)^\rho = \partial\rho * \Delta_{K^*} + \rho * K^{**}$ by Lemma 3.1 (3). The result now follows from the previous lemma.

If K is a simplex and $\dim(\tau) = 0$ the result is trivial. For the remaining cases we have

$$(K^\tau)^\sigma = \begin{cases} \partial\sigma * \Delta_{K^*} + \sigma * K^{**} & \tau = \emptyset, \sigma \neq \emptyset \\ K & \tau \neq \emptyset, \sigma = \emptyset \\ \partial\sigma * \Delta_{K^\tau} + \sigma * K & \tau \neq \emptyset, \sigma \neq \emptyset \end{cases}$$

and the result follows from the previous lemma and the case $\tau, \sigma = \emptyset$. \square

Corollary 3.7. *NH -balls are the double duals of combinatorial balls. NH -spheres are the double duals of combinatorial spheres.*

It is known that a d -homogeneous complex with $d + 2$ vertices is either the boundary of a simplex or an elementary starring of a simplex (see [12, Lemma 6]) but for a general d -dimensional complex with $d + 2$ vertices not even its homotopy type can be known beforehand. However, when the complex is an NH -manifold then it is either contractible or homotopy equivalent to a sphere. Actually, the next stronger result holds.

Proposition 3.8. *If M is an NH -manifold of dimension d and $d + 2$ vertices then M is an NH -ball or NH -sphere.*

Proof. By Lemma 3.2 $\rho = \Delta(V_M - V_{M^*}) \neq \emptyset$ and by Lemma 3.1

$$M = (M^*)^\rho = \partial\rho * \Delta_{M^*} + \rho * M^{**}.$$

If M^* is a simplex then $M = \partial\rho * \Delta_{M^*}$ is an NH -ball. Otherwise, since M is an NH -manifold then $M^{**} = lk(\rho, M)$ must be an NH -ball or NH -sphere by Theorem 2.2 (1). Therefore M is an NH -ball or an NH -sphere by Theorem 3.6. \square

4. MAIN RESULTS

In this section we generalize Dong's result on the Alexander dual of simplicial spheres [6] and Santos-Sturmfels' result on simplicial balls [15] to the more general setting of NH -spheres and NH -balls. First we need some lemmas. For $v \in V_K$, $K - v = \overline{K - st(v, K)}$ denotes the *deletion* of v .

Lemma 4.1. *Let K be a complex of dimension d and $d + 2$ vertices. Then, for every vertex $u \in V_K - V_{K^*}$ we have that $K^* = (lk(u, K))^\tau$ where $\tau = \Delta(V_K - V_{st(u, K)})$.*

Proof. By hypothesis we can write $K = \Delta^d + u * lk(u, K)$. Let τ be as in the statement. Then,

$$\begin{aligned}
\sigma \in (lk(u, K))^\tau &\Leftrightarrow \Delta(V_{lk(u, K)} \cup V_\tau - V_\sigma) \notin lk(u, K) \\
&\Leftrightarrow \Delta(V_{lk(u, K)} \cup (V_K - V_{st(u, K)}) - V_\sigma) \notin lk(u, K) \\
&\Leftrightarrow \Delta(V_K - \{u\} - V_\sigma) \notin lk(u, K) \\
&\Leftrightarrow u * \Delta(V_K - \{u\} - V_\sigma) \notin K \\
&\Leftrightarrow \Delta(V_K - V_\sigma) \notin K \\
&\Leftrightarrow \sigma \in K^*. \quad \square
\end{aligned}$$

Lemma 4.2. *Let $K \neq \Delta^d$ be a simplicial complex of dimension d and let $v \in V_K$. Then,*

- (1) $lk(v, K^*) = (K - v)^*$.
- (2) $lk(v, K) = (K^* - v)^\tau$ where $\tau = \Delta(V_{K-v} - V_{K^*-v})$.
- (3) *If v is not isolated and $lk(v, K)$ is not a simplex then $K^* - v \simeq \Sigma^t lk(v, K)^*$ for some $t \geq 0$.*
- (4) *If $lk(v, K)$ is a simplex then $K^* - v$ is contractible.*

Proof. For (1),

$$\sigma \in lk(v, K^*) \Leftrightarrow v * \sigma \in K^* \Leftrightarrow (v * \sigma)^c \notin K \Leftrightarrow \sigma^c \notin K - v \Leftrightarrow \sigma \in (K - v)^*.$$

To prove (2), take any $x \notin V_K$. Since $K \neq \Delta^d$ then $(K^x)^* = K$ and by (1),

$$lk(v, K) = lk(v, (K^x)^*) = (K^x - v)^*.$$

Note that $K^x = \Delta_K + x * K^*$, and then

$$K^x - v = \Delta_K - v + x * K^* - v = \Delta(V_K - v) + x * (K^* - v).$$

Now Lemma 4.1 implies that

$$(K^x - v)^* = lk(x, K^x - v)^\tau = (K^* - v)^\tau$$

where $\tau = \Delta(V_{K^x-v} - V_{st(x, K^x-v)}) = \Delta(V_{K-v} - V_{K^*-v})$. This proves (2).

To prove (3), apply Alexander dual to the equality given in (2) to yield

$$lk(v, K)^* = ((K^* - v)^\tau)^*.$$

When $\tau \neq \emptyset$, this equals $K^* - v$ by Lemma 3.1 (2), which settles the result with $t = 0$. Note that, by hypothesis, $K^* - v = \Delta^\tau$ and $\dim(\tau) = 0$ cannot simultaneously hold.

Suppose now that $\tau = \emptyset$. Denote $T = K^* - v$. If $\dim(T) \neq |V_T| - 2$ then $lk(v, K)^* = T^{**} = T$ by Lemma 3.2 and the result holds with $t = 0$. If $\dim(T) = |V_T| - 2$ then $\rho = \Delta(V_T - V_{T^*}) \neq \emptyset$ and

$$T = (T^*)^\rho = \partial\rho * \Delta_{T^*} + \rho * T^{**} = \partial\rho * \Delta_{T^*} + \rho * lk(v, K)^*.$$

Since by hypothesis $\Delta_{T^*} = \Delta_{lk(v, K)} \neq \emptyset$ and $T^{**} = lk(v, K)^* \neq \emptyset$ then

$$K^* - v = T \simeq \Sigma(\partial\rho * lk(v, K)^*) \simeq \Sigma^t lk(v, K)^*.$$

To prove (4) note that if $(K^* - v)^\tau = lk(v, K)$ is a simplex then $K^* - v$ is an NH -ball by Theorem 3.6. \square

The following result is standard.

Lemma 4.3. *Let K be a finite simplicial complex and $A, B \subset K$ subcomplexes such that $K = A + B$.*

- (1) If A and B are contractible then $K \simeq \Sigma(A \cap B)$. If, in addition, K is acyclic then K is contractible. In particular, acyclic simplicial complexes of dimension d and $d + 2$ vertices are contractible.
- (2) If $A \cap B$ and B are contractible then $K \simeq A$.

The following is a rewriting of [13, Lemma 2.4].

Lemma 4.4. *Let L be a subcomplex of K . Then $K \searrow L$ if and only if $K^* \nearrow L^\tau$ where $\tau = \Delta(V_K - V_L)$. In particular, if L^* is contractible or homotopy equivalent to a sphere then so is K^* .*

We are now able to give an alternative proof of Dong's and Santos-Sturmfels' original results.

Theorem 4.5 (Dong, Santos-Sturmfels). *If $B \neq \Delta^d$ is a combinatorial d -ball then B^τ is contractible. If S is a combinatorial d -sphere then S^τ is homotopy equivalent to a sphere.*

Proof. By Lemma 3.1 (4) it suffices to prove the result for $\tau = \emptyset$. We first prove it for a combinatorial ball B by induction on $d \geq 1$. If $d = 1$ then B collapses to a 1-ball with two edges (whose Alexander dual is a vertex) and the result follows from Lemma 4.4. Now, let $d \geq 2$. If $|V_B| = d + 2$, take $u \notin B^*$. If $lk(u, B)$ is not a simplex, Lemmas 4.1 and 3.1 (4) imply $B^* \simeq \Sigma^t lk(u, B)^*$, which is contractible by induction since $lk(u, B)$ is a ball. If $lk(u, B)$ is a simplex, the result follows immediately.

Suppose $|V_B| \geq d + 3$ and let $v \in \partial B$. Now, $B^* - v$ is contractible by Lemma 4.2 (4) or Lemma 4.2 (3) and induction. Since $B^* = B^* - v + st(v, B^*)$ is acyclic by Alexander duality then B^* is contractible by Lemma 4.3 (1).

Now let S be a combinatorial sphere. We may assume that $|V_S| \geq d + 3$. We proceed again by induction on d . Let $d \geq 1$ and $v \in S$. By Lemma 4.2 (1), $lk(v, S^*) = (S - v)^*$ which is contractible by Newman's theorem and the previous case. Since $S^* = S^* - v + st(v, S^*)$ where $(S^* - v) \cap st(v, S^*) = lk(v, S^*)$ is contractible, then $S \simeq S^* - v \simeq \Sigma^t lk(v, S)^*$ by Lemma 4.3 (2) and Lemma 4.2 (3). The result now follows by the inductive hypothesis on the $(d - 1)$ -sphere $lk(v, S)$. \square

Note that this theorem actually holds for simplicial (not necessarily combinatorial) balls and spheres. The more general formulation follows from this result using an argument of Dong [6], since the non-trivial cases turn out to be polytopal which, in turn, are combinatorial (see also [7, 12]).

In order to prove the generalization of Santos and Sturmfels' result, we first need to characterize the d -homogeneous subcomplex of an NH -ball of dimension d and $d + 3$ vertices. We need the following known result on manifolds with few vertices.

Theorem 4.6 ([4], Theorem A). *Let M be a boundaryless combinatorial d -manifold with n vertices. If*

$$n < 3 \left\lceil \frac{d}{2} \right\rceil + 3$$

then M is a combinatorial d -sphere. Also, if $d = 2$ and $n = 6$ then M is either PL-homeomorphic to a 2-sphere or combinatorially equivalent to the projective plane $\mathbb{R}P^2$.

The following is an immediate consequence of this result.

Corollary 4.7. *Let M be a combinatorial d -manifold with boundary with n vertices. If*

$$n < \min \left\{ 3 \left\lceil \frac{d-1}{2} \right\rceil + 3, 3 \left\lceil \frac{d}{2} \right\rceil + 2 \right\}$$

then M is a combinatorial d -ball. The result is also valid if $d = 3$ and $n = 6$.

Proof. By Theorem 4.6 ∂M is a combinatorial $(d-1)$ -sphere. This includes the case $d = 3$ and $n = 6$ since $\mathbb{R}P^2$ cannot be the boundary of a compact manifold. Take $u \notin M$ and build $N = M + u * \partial M$ where $M \cap u * \partial M = \partial M$. It is easy to see that N is a boundaryless combinatorial d -manifold. Now, since $|V_N| < 3\lceil \frac{d}{2} \rceil + 3$ then N is a combinatorial d -sphere by Theorem 4.6 and $M = \overline{N - u * \partial M}$ is a combinatorial d -ball by Newman's theorem. \square

Proposition 4.8. *Let B be an NH -ball of dimension d and $n \leq d + 3$ vertices. Then, the d -homogeneous subcomplex $Y^d \subset B$ is a combinatorial d -ball.*

Proof. Since B is acyclic, by Theorem 2.2 (4) Y^d is a weak d -pseudomanifold with boundary. We may assume $d \geq 2$ and $|V_{Y^d}| = d + 3$ since the cases $d = 0, 1$ and $|V_{Y^d}| = d + 1$ are trivial and, if $|V_{Y^d}| = d + 2$, Y^d is an elementary starring of a simplex by [12, Lemma 6]. Note that Y^d is necessarily connected. We first prove that Y^d is a combinatorial manifold. Let $v \in Y^d$. By the same reasoning as above we may assume $|V_{lk(v, B)}| = d + 2$. If $lk(v, B)$ is an NH -ball then $lk(v, Y^d)$ is a combinatorial $(d-1)$ -ball by inductive hypothesis since $lk(v, Y^d)$ is the $(d-1)$ -homogeneous part of $lk(v, B)$. Suppose $lk(v, B)$ is an NH -sphere. If $\dim_h(lk(v, B)) = d - 1$ then $lk(v, B) = lk(v, Y^d)$ is a combinatorial $(d-1)$ -sphere by Proposition 2.4. Otherwise, $lk(v, Y^d)$ is the $(d-1)$ -homogeneous part of the NH -ball in any decomposition of $lk(v, B)$ and the result follows again by induction. This shows that Y^d is a combinatorial d -manifold.

Suppose $d = 2$. Note that Y^d is \mathbb{Z}_2 -acyclic since it is connected, it has non-empty boundary and it is contained in the acyclic complex B . On the other hand, any \mathbb{Z}_2 -acyclic complex with 5 vertices is collapsible (see for example [1, Theorem 1]).

For $d \geq 3$, Y^d is a combinatorial d -ball by Corollary 4.7. \square

Proposition 4.9. *Any NH -ball B of dimension $d \geq 2$ and $d + 3$ vertices collapses to a complex of dimension $d - 2$.*

Proof. We first show that all the principal $(d-1)$ -simplices in B can be collapsed. Let Y^{d-1} be the subcomplex of B generated by the principal $(d-1)$ -simplices and let Y^d be the d -homogeneous part of B . By the previous proposition, Y^d is a combinatorial ball. Suppose that not all the $(d-1)$ -simplices in Y^{d-1} can be collapsed. Let K be the subcomplex of Y^{d-1} generated by these $(d-1)$ -simplices. By assumption, $K \neq \emptyset$. Note that K is a weak $(d-1)$ -pseudomanifold with boundary by Theorem 2.2 (4) but it has no free $(d-2)$ -faces in B . Then $\partial K \subset Y^d$. Therefore, if c denotes the formal sum of the $(d-1)$ -simplices of K then $c \in H_{d-1}(B, Y^d)$. Since B and Y^d are contractible then $H_{d-1}(B, Y^d) = 0$. This implies that c is not a generating cycle, which is a contradiction since the $(d-1)$ -simplices of c are maximal. This shows that we can collapse all the principal $(d-1)$ -simplices in B . On the other hand, since Y^d is a combinatorial d -ball with $d + 3$ vertices or less, it is vertex decomposable by [10, 5.7]. In particular Y^d is collapsible with no need of further subdivision. Then we can make the collapses in order of decreasing dimension and collapse the d -simplices and the $(d-1)$ -simplices of Y^d afterwards to obtain a $(d-2)$ -dimensional complex. \square

Corollary 4.10. *Any NH -ball of dimension $d \geq 3$ and $d + 2$ vertices collapses to a complex of dimension $d - 3$.*

Proof. We proceed by induction on d . If $d = 3$ then B is collapsible since it is acyclic and has few vertices (see [1, Theorem 1]). Let $d \geq 4$ and write $B = \Delta^d + st(u, B)$ where $u \notin \Delta^d$. Now, $\Delta^d \cap st(u, B) = lk(u, B) \subset \Delta^d$ is an NH -ball since B is one. Also, $\dim(lk(u, B)) \leq d - 1$ and $|V_{lk(u, B)}| \leq d + 1$. Let $m = |V_{lk(u, B)}| - \dim(lk(u, B))$. If $m = 1$ then $lk(u, B)$ is a simplex and $B \searrow \Delta^d \searrow 0$. For $m = 2, 3, 4$ we use the inductive hypothesis, Proposition 4.9 or Corollary 2.5 respectively to show that $lk(u, B)$ collapses to a complex of dimension $\dim(lk(u, B)) - (5 - m) = |V_{lk(u, B)}| - m - (5 - m) = |V_{lk(u, B)}| - 5 \leq d + 1 - 5 = d - 4$. Therefore, $u * lk(u, B) = st(u, B)$ collapses to a complex of dimension $d - 3$. Finally, if $m \geq 5$ then $\dim(lk(u, B)) \leq |V_{lk(u, B)}| - 5 \leq d - 4$ and $\dim(st(u, B)) \leq d - 3$. In any case we can collapse afterwards the i -simplices of Δ^d ($i = d, d - 1, d - 2$) in order of decreasing dimension to obtain a $(d - 3)$ -dimensional complex. \square

We are ready to prove now the first of our main results.

Theorem 4.11. *Let B be an NH -ball and let τ be a simplex (possibly empty). Then, B^τ is contractible.*

Proof. By Lemma 3.1 (4) we only need to prove the case $\tau = \emptyset$. It suffices to prove that B^* is simply connected. Let $d = \dim(B)$ and $n = |V_B|$. We can assume that $d \geq 2$ since in lower dimensions all NH -balls are combinatorial. We can also assume that $2 \leq n - d \leq 4$, since if $n - d \geq 5$, a simple argument of Dong [6] shows that B^* is simply connected (it contains the complete 2-skeleton of $\Delta(V_{B^*})$).

If $n \leq 7$, B^* is collapsible since it is acyclic and it has few vertices ([1, Theorem 1]). For $n \geq 8$, by Proposition 4.9 and Corollaries 2.5 and 4.10 there exists a subcomplex $K \subset B$ such that $B \searrow K$ with $V_K = V_B$ and $|V_K| - \dim(K) = 5$. Therefore $B^* \nearrow K^*$, and since $|V_K| - \dim(K) = 5$, K^* is simply connected. \square

Our next goal is to prove the second of our main results.

Theorem 4.12. *Let S be an NH -sphere and let τ be a simplex (possibly empty). Then, S^τ is homotopy equivalent to a sphere.*

Like in the proof for NH -balls, we only need to prove the case $\tau = \emptyset$ and $2 \leq |V_S| - \dim(S) \leq 4$ since, as before, if $|V_S| - \dim(S) \geq 5$, then S^* is simply connected, and a simply connected space with the homology of a sphere is homotopy equivalent to one. We can suppose also that $\dim_h(S) < \dim(S)$ by Proposition 2.4 and Theorem 4.5. The 1-dimensional case is easy to verify.

The proof of Theorem 4.12 will be divided in the following four cases. Let $d = \dim(S) \geq 2$, $n = |V_S|$ and $k = \dim_h(S)$. We handle each case separately.

- (A) $n = d + 2$ and $k = d - 1$.
- (B) $n = d + 2$ and $k = d - 2$.
- (C) $n = d + 3$ and $k = d - 1$.
- (D) Remaining cases.

Proof of Case (D). We will show that $S \searrow K$ with $|V_K| - \dim(K) = 5$. The result will follow immediately from Lemma 4.4 and the fact that K^* is simply connected. The case $n = d + 4$ follows directly from Corollary 2.5 by collapsing (only) the d -simplices of S .

Suppose now that $n = d + 2$ or $d + 3$ and let $S = B + L$ be a decomposition. We first analyze the case $n = 5$. In this situation, $L = *$. If $d = 2$ then B is acyclic with four vertices and if $d = 3$ then $B = \Delta^3$. Similarly as in the 1-dimensional case, $S \searrow S^0$ and the result follows from Lemma 4.4.

Suppose $n = d + 3$ with $n \geq 6$. The complex B in the decomposition of S is an NH -ball of dimension d and $|V_B| \in \{d + 1, d + 2, d + 3\}$. In any case, B collapses to a $(d - 2)$ -dimensional complex T whether because $B = \Delta^d$ or by Corollary 4.10 or Proposition 4.9. Moreover, since $d \geq 3$, we can arrange the collapses in order of decreasing dimension to get $V_T = V_B$ by collapsing only the d and $(d - 1)$ -dimensional simplices. Since $\dim(L) \leq d - 2$ and it is top generated, the collapses in $B \searrow T$ can be carried out in S and therefore $S \searrow K = T + L$, which is a complex with the desired properties.

The case $n = d + 2$ with $n \geq 6$ follows similarly as the previous case by showing that S collapses to a $(d - 3)$ -dimensional complex with the same vertices. \square

Proof of Case (A). We proceed by induction. Write $S = \Delta^d + u * lk(u, S)$ with $u \notin \Delta^d$. Note that $lk(u, S)$ is an NH -sphere of homotopy dimension $d - 2$ and dimension $d - 2$ or $d - 1$. By Lemmas 4.1 and 3.1 (4) it suffices to show that $lk(u, S)^*$ is homotopy equivalent to a sphere. If $\dim(lk(u, S)) = d - 2$ then $lk(u, S)$ is homogeneous by Proposition 2.4 and the result follows from Theorem 4.5. If $\dim(lk(u, S)) = d - 1$ then $|V_{lk(u, S)}| = d + 1$ and the result follows by the inductive hypothesis. \square

In order to prove the cases (B) and (C) we need some preliminary results.

Lemma 4.13. *Let $S = B + L$ be a decomposition of an NH -sphere. If $v \in L$ then $S - v$ is contractible.*

Proof. If $v \in L^\circ$ then $L - v$ deformation retracts to $\partial L \subset B$ and, hence, $S - v \simeq B \simeq *$. Otherwise, $v \in \partial L \cap \tilde{\partial} B$ and $S - v = (B - v) + (L - v)$ with $(B - v) \cap (L - v) = \partial L - v$. Since $v \in \partial L \cap \tilde{\partial} B$, then $B - v$ and $L - v$ are contractible. On the other hand, $\partial L - v$ is contractible by Newman's theorem. Hence, $S - v$ is contractible. \square

Lemma 4.14. *Let $S = B + L$ be a decomposition of an NH -sphere of dimension $d \geq 1$ satisfying the hypotheses of case (C). If $lk(v, S)$ is a combinatorial $(d - 2)$ -sphere then $S - v$ is an NH -ball.*

Proof. We proceed by induction in d . The case $d = 1$ is straightforward. Let $d \geq 2$. We prove first that $S - v$ is an NH -manifold.

Let $w \in S - v$. We have to show that its link is an NH -sphere or an NH -ball. If $w \notin st(v, S)$ then $lk(w, S - v) = lk(w, S)$ which is an NH -ball or NH -sphere. Suppose $w \in st(v, S)$. We will show first that $lk(w, S)$ is an NH -sphere of homotopy dimension $d - 2$. We prove this in various steps. Note that this is clear if $w \in L$, so we may suppose $w \notin L$.

Step 1. We first prove that if $v \notin L$ then there is a d -simplex in $st(w, S)$ which is adjacent to a $(d - 1)$ -simplex of L . Write $\Delta^d = \{v, w\}^c$. Since $v, w \notin L$ then $L \subset \Delta^d$ and therefore $\Delta^d \not\subset S$ because L is top generated in S . Since $\dim(S) = d$ and $st(v, S)$ is $(d - 1)$ -homogeneous then w is a face of d -simplex ρ not containing v . Since any two $(d - 1)$ -faces of Δ^d are adjacent then ρ is adjacent to some $(d - 1)$ -simplex of L .

Step 2. We now prove that the inclusion induces an isomorphism $H_{d-1}(S - w) \simeq H_{d-1}(B - w)$. On one hand, the induced homomorphism $H_{d-1}(B - w) \rightarrow H_{d-1}(S - w)$ is injective since $(S - w) - (B - w) = L - w$ is $(d - 1)$ -dimensional. To prove that it is

also surjective we show that any $(d-1)$ -cycle in $S-w$ cannot contain a $(d-1)$ -simplex of L . Suppose $\sigma \in L$ is a non-trivial factor in a $(d-1)$ -cycle c^{d-1} of $S-w$. Then every $(d-1)$ -simplex in L appears in c^{d-1} since c^{d-1} is a cycle and L is a top generated combinatorial $(d-1)$ -ball. If $v \in L$ then every $(d-1)$ -simplex of $st(v, S)$ appears in c^{d-1} since $st(v, S)$ is also a top generated $(d-1)$ -ball. In this case, at least one $(d-1)$ -simplex of $st(v, S)$ belongs to $st(w, S)$, contradicting the fact that c^{d-1} is a cycle in $S-w$. On the other hand, if $v \notin L$ then there exists by step 1 a principal $(d-1)$ -simplex $\tau \in L$ with a boundary $(d-2)$ -face $\eta < \rho \in st(w, S)$ with $\dim(\rho) = d$. Let $z = lk(\eta, \tau)$. Note that there are no d -simplices outside $st(w, S)$ containing η since neither v, w nor z may belong to such d -simplex and $|V_S| = d+3$. Since S is an NH -manifold, τ is the only principal $(d-1)$ -simplex containing η , and then $\partial c^{d-1} \neq 0$ in $S-w$, which is a contradiction.

Step 3. We prove that $lk(w, S)$ is an NH -sphere of homotopy dimension $d-2$. We claim first that $H_{d-1}(S-w) = 0$. By step 2 it suffices to show that $H_{d-1}(B-w) = 0$. From the Mayer-Vietoris sequence applied to $B = B-w + st(w, S)$ and the fact that $lk(w, B) = lk(w, S)$ (here we use that $w \notin L$), it follows that $H_{d-1}(B-w) \simeq H_{d-1}(lk(w, S))$. If $H_{d-1}(lk(w, S)) \neq 0$ then $lk(w, S)$ is $(d-1)$ -homogeneous by Proposition 2.4, which is a contradiction since $st(w, S)$ contains at least a $(d-1)$ -simplex. Thus, the claim is proved.

If we now consider the Mayer-Vietoris sequence for $S = S-w + st(w, S)$ in degree $d-1$ one has that $\mathbb{Z} \simeq H_{d-1}(S) \rightarrow H_{d-2}(lk(w, S))$ is injective, so $H_{d-2}(lk(w, S)) \neq 0$ and therefore $lk(w, S)$ is an NH -sphere of homotopy dimension $d-2$.

Finally if $\dim(lk(w, S)) = d-2$ then $lk(w, S)$ is a combinatorial $(d-2)$ -sphere by Proposition 2.4 and therefore, $lk(w, S-v) = lk(w, S) - v$ is a combinatorial $(d-2)$ -ball by Newman's theorem. Suppose that $\dim(lk(w, S)) = d-1$. If $|V_{lk(w, S)}| = d+1$ then we may write $lk(w, S) = \Delta^{d-1} + st(v, lk(w, S))$ since v is not a vertex of a d -simplex in S . In this case $lk(w, S) - v = \Delta^{d-1}$. If $|V_{lk(w, S)}| = d+2$ we may apply the inductive hypothesis since $lk(v, lk(w, S)) = lk(w, lk(v, S))$ is a combinatorial $(d-3)$ -sphere, and conclude that $lk(w, S) - v$ is an NH -ball. This proves that $S-v$ is an NH -manifold.

We prove now that $S-v$ is an NH -ball. Note that $\dim(S-v) = d$ and $|V_{S-v}| = d+2$, so by Proposition 3.8 we only need to prove that it is acyclic, and this follows immediately from the Mayer-Vietoris sequence applied to $S = S-v + st(v, S)$. \square

Lemma 4.15. *Let $S = B + L$ be a decomposition of an NH -sphere satisfying the hypotheses of case (C). If there is a vertex v in L such that $\dim(S-v) = d$ and there is a non-edge $\{u, w\}$ of S with $u, w \neq v$ then $(S-v)^*$ is contractible.*

Proof. By Lemma 3.2, $|V_{S^*}| = d+3$. By hypothesis $\{u, w\}^c \in S^*$ is a d -simplex and since $v \neq u, w$ then $v \in \{u, w\}^c$. Therefore, $\dim(lk(v, S^*)) = d-1$. On the other hand, there exists a d -simplex $\eta \in S$ with $v \notin \eta$; hence $\{v, a\} := \eta^c \notin S^*$. Therefore, $|V_{lk(v, S^*)}| \leq d+1$. If $|V_{lk(v, S^*)}| = d$ then $(S-v)^* = lk(v, S^*)$ is a $(d-1)$ -simplex. If $|V_{lk(v, S^*)}| = d+1$ then $lk(v, S^*) = (S-v)^*$ is acyclic by Lemma 4.13 and Alexander duality, and therefore contractible by Lemma 4.3 (1). \square

Lemma 4.16. *Let S be an NH -sphere satisfying the hypotheses of case (C). Then, for any decomposition $S = B + L$ there exists $z \in V_L$ such that $(S-z)^*$ is contractible.*

Proof. We proceed by induction in d . The 1-dimensional case is straightforward. Let $d \geq 2$ and let $u \in V_L$. If $\dim(lk(u, S)) = d-2$ then $lk(u, S)$ is a combinatorial $(d-2)$ -sphere and the result follows from Lemma 4.14 and Theorem 4.11. Suppose $\dim(lk(u, S)) = d-1$. We analyze the two possible cases $|V_{lk(u, S)}| = d+1$ or $|V_{lk(u, S)}| = d+2$.

If $|V_{lk(u,S)}| = d + 1$, let $w \in S$ such that $\{u, w\} \notin S$. Let Δ^d be a d -simplex containing u and let $v = V_S - V_{\Delta^d} - \{w\}$. Since L is top generated then either $v \in L$ or $w \in L$. If $v \in L$ then Lemma 4.15 implies that $(S - v)^* \simeq *$. Assume then that $v \notin L$ (and hence $w \in L$). We may assume $\dim(lk(w, S)) = d - 1$ since otherwise w is the desired vertex by Lemma 4.14 and Theorem 4.11 again. Let $\tilde{\Delta}^d$ be a d -simplex containing w . Since L is top generated, $w \in L$ and $v \notin L$ then $\tilde{\Delta}^d = w * v * \Delta^{d-2}$ with $\Delta^{d-2} \prec \Delta^d - u$. Let $x \neq u$ be the only vertex in Δ^{d-2} not in $\tilde{\Delta}^d$. Then, $x \in L$ and it fulfils the hypotheses of Lemma 4.15. Therefore $(S - x)^*$ is contractible.

Suppose finally that $|V_{lk(u,S)}| = d + 2$. From the decomposition $lk(u, S) = lk(u, B) + lk(u, L)$ there exists $y \in lk(u, L)$ such that $(lk(u, S) - y)^* \simeq *$ by the inductive hypothesis. If $u \notin (S - y)^*$ then $u^c \in S - y$; i.e. $S - y - u = \Delta^d$. In this case, we can write $S - y = \Delta^d + u * lk(u, S - y)$ and we have $(S - y)^* = (lk(u, S) - y)^\tau$ by Lemma 4.1. If $lk(u, S) - y$ is not a simplex then $(lk(u, S) - y)^\tau \simeq \Sigma^t(lk(u, S) - y)^* \simeq *$ by Lemma 3.1 (4) and if $lk(u, S) - y = \Delta^r$ then $\tau \neq \emptyset$ and $(lk(u, S) - y)^\tau = \partial\tau * \Delta^r \simeq *$. In either case, y is the desired vertex. Assume $u \in (S - y)^*$. Then we have a non-trivial decomposition

$$(S - y)^* = (S - y)^* - u \underset{lk(u, (S - y)^*)}{+} st(u, (S - y)^*).$$

Since neither $S - y$ nor $lk(u, S) - y$ are simplices and $u \in S - y$ is not isolated then $(S - y)^* - u \simeq \Sigma^t lk(u, S - y)^* \simeq *$ by Lemma 4.2 (3). The result then follows by Lemmas 4.3 (1) and 4.13. \square

Proof of Cases (B) and (C). We prove (B) and (C) together by induction in d . Let $S = B + L$ be a decomposition.

If $d = 2$, B is collapsible since it is acyclic and has few vertices. Then $S \searrow S^0$ for (B) and $S \searrow S^1$ for (C). The results then follow in both cases from Lemma 4.4.

Let $d \geq 3$. Suppose first that S satisfies the hypotheses of (B). Write $S = \Delta^d + v * lk(v, S)$. Then $S^* = lk(v, S)^\tau$ for $\tau = \Delta(V_S - V_{st(v, S)})$ by Lemma 4.1. Since $lk(v, S)$ is an NH -sphere of dimension $\leq d - 1$ then the result follows from Theorem 4.5, cases (A) and (D) or the inductive hypothesis on (B) and (C).

Finally suppose S satisfies the hypotheses of (C). By Lemma 4.16 there exists $v \in V_L$ such that $(S - v)^* \simeq *$. Write $S^* = S^* - v + st(v, S^*)$ where $(S^* - v) \cap st(v, S^*) = lk(v, S^*) = (S - v)^* \simeq *$. By Lemmas 4.3 (2) and 4.2 (3), $S^* \simeq S^* - v \simeq \Sigma^t lk(v, S)^*$ (note that v is not isolated nor $lk(v, S)$ is a simplex because $v \in L$). Since $lk(v, S)$ is an NH -sphere of dimension $\leq d - 1$ then $lk(v, S)^*$ is homotopy equivalent to a sphere by Theorem 4.5, cases (A) and (D) or inductive hypothesis on (B) and (C). \square

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